Trigonometric Identities and exact values.

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LEVEL

High School, once students have a sound understanding of the Unit Circle. It would be an advantage (but not totally necessary) to have done the activity called *Sine and cosine functions and exact values*.

OBJECTIVES

(i) To introduce students to some basic trigonometric identities in an interesting way.
(ii) To see how trigonometric identities help to answer some questions posed by inquisitive mathematical minds.
(iii) To use a calculator to provide early confidence and lead students to being able to use previously learned skills (surd manipulation) in a powerful and meaningful way.

OVERVIEW

Traditionally, trigonometric identities have been something to learn and something to apply to fairly meaningless questions. This activity offers an alternative for a couple of identities. This activity could be used as a way to introduce the idea of identities and then could be built on later.

EXPLORATORY ACTIVITIES

[Note]
(a) We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu, using [SHIFT] [MODE] [1] (MthIO)

Trigonometric Identities and exact values.

Activity 1: A simple identity.

We can easily find the value of, for example, cos(30^o) using the *991ES*.in *general calculation* mode. Do the following:

[Operations]

- Enter the general calculation mode — [MODE] [1]
- Set the calculator to read angle values as degrees. — [SHIFT] [MODE] (SETUP) [3]
- Enter cos(30) — [cos] [3] [0] [)]
- Press = — [=]

This should reveal the following:

Note that, in this case, the *991ES* returns the value in *exact form* (or in terms of a surd). You should know (of by heart) the exact form of the values of the sine, cosine and tangent for $0^o, 30^o, 45^o, 60^o$ and 90^o and the *sister* values in other quadrants of the unit circle.

Use the *991ES* to complete the following table:

θ	$\cos\theta$	$\sin\theta$
15		
75		
36		
54		
18		
72		

From your table, you should be able to see something intriguing. This illustrates one of the simplest general truths about trigonometric functions. General truths of this type are often called *identities*. This identity can be written as follows:

$$\cos\theta = \sin(90 - \theta)$$

The proof of this can be established by consider the unit circle and symmetry.

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Activity 2: The addition formulae

Check that the only values of θ ($0^o \le \theta \le 90^o$, in 5^o increments) that the *991ES* reports the cosine of in exact (surd) form, are: $0^o, 15^o, 30^o, 45^o, 60^o, 75^o$ and 90^o. Record the values of each.

Notice that:

- All are multiples of 15^o and as a result
- 45 − 30 = 15
- 90 − 15 = 75
- 60 + 15 = 75
- 45 + 30 = 60

and so on.

This might suggest that there is a way to calculate $\cos(15)$ if you knew $\cos(45)$ and $\cos(30)$.

In fact there is a way. A series of trigonometric identities exist called the addition formulae. Two of these are:

- $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
- $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

You should look into how these formulae can be derived.

We can use these in the following way:

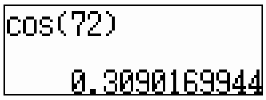
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\Rightarrow \quad \cos(45^o - 30^o) = \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$$
$$\Rightarrow \quad \cos(15^o) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$
$$\Rightarrow \quad \cos(15^o) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$
$$\Rightarrow \quad \cos(15^o) = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$\Rightarrow \quad \cos(15^o) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

This result is exactly as given on the *991ES*.

Activity 3: Investigating seventy-two degrees.
The *991ES* does not return exact values for any other angles other than the multiples of 15^o. However, an inquisitive mathematical mind would now start to wonder if there are any more integer angles for which the exact value of its cosine and sine could be found.

Much effort was spent on such things over many years by mathematicians of the past. We will now look at some of the results of such work.

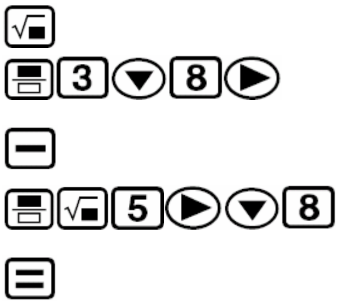
Use the *991ES* to determine a decimal approximation for $\cos 72^o$.



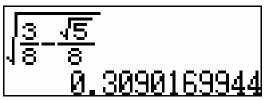
Now find a decimal approximation for the following: $\sqrt{\frac{3}{8}-\frac{\sqrt{5}}{8}}$. To do this use the following steps

[Operations]

- Enter the square root
- Enter the fraction $\frac{3}{8}$
- Enter the subtraction sign
- Enter the fraction $\frac{\sqrt{5}}{8}$
- Press =



You should achieve the following:

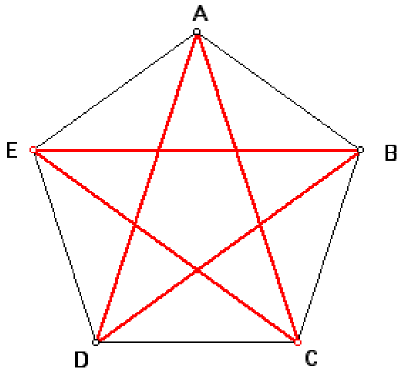


So comparing the decimal approximation for $\cos 72^o$ and $\sqrt{\frac{3}{8}-\frac{\sqrt{5}}{8}}$, we would wonder if $\cos 72^o = \sqrt{\frac{3}{8}-\frac{\sqrt{5}}{8}}$.

This is, in fact, true. Consider the following proof.

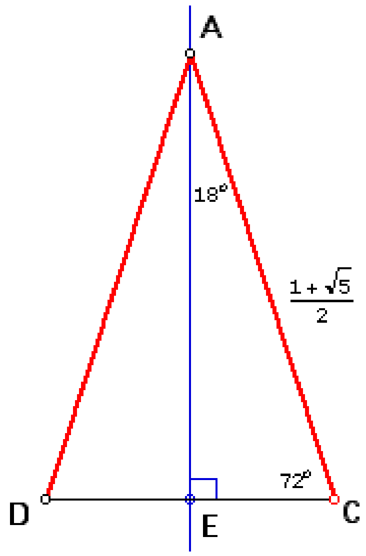
In a regular pentagon of side length 1 unit, the diagonals (in red) have length phi (the golden ratio). Phi has exact value $\frac{1+\sqrt{5}}{2}$.

Note also that angle BCA is 36^o; using the fact that triangle BCA is isosceles and a regular pentagon has interior angles of 105^o



So, in triangle ADC we can proceed as follows:

AE is drawn perpendicular to DC and since triangle is isosceles, E is the midpoint of DC. Angle ACE is (angle BCA is 36^o) and hence angle EAC is 18^o.



DAC
72^o

Now, $\cos(72^o) = \frac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}}{4} - \frac{1}{4}$

You should be able to show that:
$\frac{\sqrt{5}}{4} - \frac{1}{4} = \sqrt{\frac{3}{8} - \frac{\sqrt{5}}{8}}$

Activity 4: Halving.

One of the quests for mathematicians over the years has been to see if they could find the exact values for the cosine and sine of certain *fractions of angles* for which they already knew the exact values for; for example 72^o, could they find the exact values of cosine and sine of $\frac{1}{3}$ of 72^o and then for $\frac{1}{3}$ of this result and so on?

What about $\frac{1}{2}$? Obvious things were tried like just halving the cosine value, but this did not work.

Consider the following trigonometric identity: $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$

You should research to find how this could be derived. It is called *a half angle formula*.

We could apply this identity as follows:

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\Rightarrow \quad \cos\left(\frac{72^0}{2}\right) = \sqrt{\frac{1+\cos 72^o}{2}}$$

$$\Rightarrow \quad \cos\left(36^o\right) = \sqrt{\frac{1+\cos 72^o}{2}}$$

$$\Rightarrow \quad \cos\left(36^o\right) = \sqrt{\frac{1+\frac{\sqrt{5}}{4}-\frac{1}{4}}{2}}$$

$$\Rightarrow \quad \cos\left(36^o\right) = \sqrt{\frac{3}{8}+\frac{\sqrt{5}}{8}}$$

$$\Rightarrow \quad \cos\left(36^o\right) = \sqrt{\frac{3+\sqrt{5}}{8}}$$

Use the *991ES* to check this result.

If you are interested in think more about the sorts of ideas presented in this document you should visit the following excellent website:
http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/simpleTrig.html#exactrig

You will find some simply wonderful mathematics here, especially the section on the patterns that are formed by exact values of cosines and sines.

EXERCISES

The purpose of the exercises is to give you an opportunity to do some independent work, to develop fluency with calculator use and to learn more about circular and angular measurements.

Exercise 1.

Use a half angle formula to find the exact value of $\cos 18^o$. Use you calculator to gain a decimal approximation for $\cos 18^o$ to check your answer.

Exercise 2

Use a triangle within a regular pentagon of side length 1 unit to find the exact value of $\cos 18^o$.

Exercise 3

Use an addition formula to show that $\cos(54^o) = \sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$.

SOLUTIONS

Table from Activity 1.

θ	$\cos\theta$	$\sin\theta$
15	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$
75	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
36	0.809	0.588
54	0.588	0.809
18	0.951	0.309
72	0.309	0.951

Exercise 1.

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\Rightarrow \quad \cos\left(\frac{36^0}{2}\right) = \sqrt{\frac{1+\cos 36^o}{2}}$$

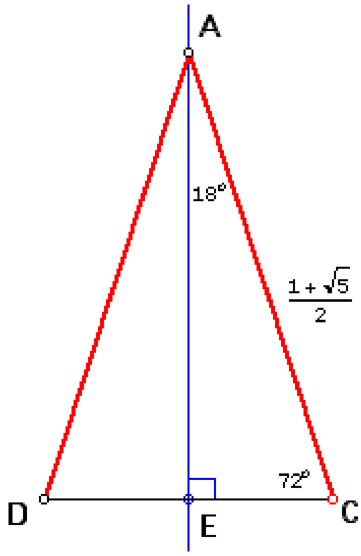
$$\Rightarrow \quad \cos\left(18^o\right) = \sqrt{\frac{1+\cos 36^o}{2}}$$

$$\Rightarrow \quad \cos\left(18^o\right) = \sqrt{\frac{1+\sqrt{\frac{3+\sqrt{5}}{8}}}{2}}$$

$$\Rightarrow \quad \cos\left(18^o\right) = \sqrt{\frac{1}{2}+\sqrt{\frac{3+\sqrt{5}}{32}}}$$

Exercise 2

Using the same triangle as earlier we can proceed as follows:



$$AE = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{4}}$$

$$\Rightarrow \quad AE = \sqrt{\left(\frac{1+2\sqrt{5}+5}{4}\right) - \frac{1}{4}}$$

$$\Rightarrow \quad AE = \sqrt{\frac{2\sqrt{5}+5}{4}}$$

Hence

$$\cos 18^o = \frac{\sqrt{\frac{2\sqrt{5}+5}{4}}}{\frac{1+\sqrt{5}}{2}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{2\sqrt{5}+5}{4}} \times \frac{2}{1+\sqrt{5}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{8\sqrt{5}+20}{4(1+\sqrt{5})^2}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{2\sqrt{5}+5}{6+2\sqrt{5}}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{2\sqrt{5}+5}{6+2\sqrt{5}} \times \frac{6-2\sqrt{5}}{6-2\sqrt{5}}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{12\sqrt{5}-20+30-10\sqrt{5}}{16}}$$

$$\Rightarrow \quad \cos 18^o = \sqrt{\frac{2\sqrt{5}+10}{16}}$$

$$\Rightarrow \quad \cos 18^o = \frac{\sqrt{2\sqrt{5}+10}}{4}$$

Note that Exercise 1 returned the result $\cos(18^o) = \sqrt{\frac{1}{2} + \sqrt{\frac{3+\sqrt{5}}{32}}}$ and the solution to Exercise 2 differs in form. Are they same number? You might like to find out by using either the *991ES* or by applying the rules of surds.

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Exercise 3

First we note that 54 = 72 – 18. Also we note that since $\cos\theta = \sin(90 - \theta)$, we can say that $\cos 18^o = \sin(90 - 18^o) = \sin(72^o)$.

Then we can proceed as follows:

$$\cos(54^o) = \cos(72^o - 18^o)$$

$$\Rightarrow \cos(54^o) = \cos 72^o \cos 18^o + \sin 72^o \sin 18^o$$

$$\Rightarrow \cos(54^o) = \left(\frac{\sqrt{5}}{4} - \frac{1}{4}\right)\left(\frac{\sqrt{2\sqrt{5}+10}}{4}\right) + \left(\frac{\sqrt{2\sqrt{5}+10}}{4}\right)\left(\frac{\sqrt{5}}{4} - \frac{1}{4}\right)$$

$$\Rightarrow \cos(54^o) = 2\left(\frac{\sqrt{5}}{4} - \frac{1}{4}\right)\left(\frac{\sqrt{2\sqrt{5}+10}}{4}\right)$$

$$\Rightarrow \cos(54^o) = \frac{1}{8}\left(\sqrt{5} - 1\right)\left(\sqrt{2\sqrt{5}+10}\right)$$

$$\Rightarrow \cos(54^o) = \frac{1}{8}\left(\sqrt{\left(\sqrt{5} - 1\right)^2}\right)\left(\sqrt{2\sqrt{5}+10}\right)$$

$$\Rightarrow \cos(54^o) = \frac{1}{8}\left(\sqrt{-2\sqrt{5}+6}\right)\left(\sqrt{2\sqrt{5}+10}\right)$$

$$\Rightarrow \cos(54^o) = \frac{1}{8}\sqrt{-20 - 20\sqrt{5} + 12\sqrt{5} + 60}$$

$$\Rightarrow \cos(54^o) = \frac{1}{8}\sqrt{40 - 8\sqrt{5}}$$

$$\Rightarrow \cos(54^o) = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$$

