# Solving Logarithms Problems

**Mun Chou, Fong**
QED Education Scientific
Malaysia

## LEVEL
High school after students have learned exponential and logarithmic functions.

## OBJECTIVES
(i) To investigate algebraic properties of logarithms.
(ii) To solve problems using these properties with the aid of the calculator.

## OVERVIEW
Logarithmic function is transcendental and do not observe the usual arithmetic algebra of polynomial function. In this exploration we explore each algebraic property of logarithm with example. We also look at using these properties to solve logarithms problems.

## EXPLORATORY ACTIVITIES
[Note]
(a) We shall use small letter $x$, $y$, $a$ and $b$ instead of capitals X, Y , A and B as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu by tapping

[SHIFT] [MODE] [1] (MthIO)

In the study of index and exponential function, we know that $a^2 \times a^3$ is equivalent to $a^5$. And in general we do can write

$$a^m \times a^n = a^{m+n}$$

Does similar property exist in logarithm?

**Example 1**
Find (i) $\log_5 3 + \log_5 2$ and (ii) $\log_5 (3 \times 2)$ and compare the answers.
**Solutions**
By using the calculator to help us find both (i) and (ii),

**[Operations]**

- Go to Computation mode — [MODE] [1]
- Find addition of $\log_5 3$ — [log▪□] [5] [▶] [3] [▶] [+]
- And $\log_5 2$ — [log▪□] [5] [▶] [2]
- Press to solve — [=]

We have the answer of $\log_5 3 + \log_5 2$ as 1.113282753. Now calculate (ii).

**[Operations]**

- Enter to find $\log_5 (3 \times 2)$ — [log▪□] [5] [▶] [3] [×] [2]
- Press to solve — [=]

So it seems like $\log_5 (3 \times 2)$ is equivalent to 1.113282753. Is it possible that $\log_5 3 + \log_5 2$ = $\log_5 (3 \times 2)$ ? Let's do another more descriptive example.

**<u>Example 2</u>**

Use the 'Calc' feature to show that ( $\log_{10} A + \log_{10} B$ ) - $\log_{10} (A \times B)$ is equal to 0 for
(i) *A* = 6, *B* = 7, (ii) *A* = 50, *B* = 33, (iii) *A* = -3, *B* = 10

**<u>Solutions</u>**

**[Operations]**

- Go to Computation mode — [MODE] [1]
- Enter ( $\log_{10} A + \log_{10} B$ ) — [(] [log] [ALPHA] [(−)] [)] [+] [log] [ALPHA]
- Continue entering — [°'''] [)] [)] [−]
- Then enter $\log_{10} (A \times B)$ — [log] [ALPHA] [(−)] [×] [ALPHA] [°'''] [)]
- Press to calculate — [CALC]
- Enter *A* = 6, *B* = 7 when prompted — [6] [=] [7] [=]

The result 0 is displayed. Calculate with the values in (ii).

- Press to calculate — [CALC]
- Enter *A* = 50, *B* = 33 — [5] [0] [=] [3] [3] [=]

Again we have the result of 0 displayed. Finally calculate with the values in (iii).

- Calculate for *A* = -3 and *B* = 10 — [CALC] [(−)] [3] [=] [1] [0] [=]

An error message is displayed. What does this imply then? From these two examples, we have explored more about the logarithm of a product. In general we can state it as follow.

♦ *For A > 0 and B > 0,* $\log_a(A \times B) = \log_a A + \log_a B$.

While calculating with values in (iii), the return result is an error message. This is due to the fact that a negative value was inputted for *A*, contrary to the definition of logarithm.

**Logarithm of a Quotient**

It is true that $a^4 \div a^3$ is equivalent to $a^{4-3}$, and in general $a^m \div a^n$ is equivalent to $a^{m-n}$. As the previous discussion shown, is it also possible that the logarithm of a quotient might satisfy the property

$$\log_a \frac{A}{B} = \log_a A - \log_a B\,?$$

**<u>Example 3</u>**

Use the 'Calc' feature to show that $(\ln A - \ln B) - \ln\frac{A}{B}$ is equal to 0 for

(i) *A* = 15, *B* = 300, (ii) *A* = 10, *B* = 9 (iii) *A* = 6, *B* = 0

**<u>Solutions</u>**

**[Operations]**

- Go to Computation mode — [MODE] [1]
- Enter $(\ln A - \ln B)$ — [(] [ln] [ALPHA] [(−)] [)] [−] [ln] [ALPHA] [°'"] [)] [)]
- Continue entering — [−]
- Then enter $\ln\frac{A}{B}$ — [ln] [▭/▭] [ALPHA] [(−)] [▼] [ALPHA] [°'"]
- Press to calculate — [CALC]
- Enter *A* = 15, *B* = 300 as prompted — [1] [5] [=] [3] [0] [0] [=]

As expected the result 0 is displayed. Now calculate with the values in (ii).

- Calculate for *A* = 10, *B* = 9 — [CALC] [1] [0] [=] [9] [=]

Again we have the result of 0 displayed.

- Calculate for *A* = 6 and *B* = 0 — [CALC] [6] [=] [0] [=]

An error message is displayed when we enter the set of value of (iii). This is in line with the exploration on the logarithm of product above; *B* should be > 0 . In general we can state this logarithm of quotient as follow.

♦ *For A > 0 and B > 0,* $\log_a \frac{A}{B} = \log_a A - \log_a B$.

 

## Other Properties of Logarithms

### (i) Power Rule for Logarithms

Let's begin with this exercise: Evaluate (i) $\log_{10} 100$ and (ii) $\log_4 64$ in the calculator.

**[Operations]**

- Go to Computation mode — [MODE] [1]
- First find $\log_{10} 100$ — [log] [1] [0] [0]
- Press to calculate — [=]

The calculator returns $\log_{10} 100$ =2. Now find $\log_4 64$.

**[Operations]**

- Enter $\log_4 64$ — [log▪□] [4] [▶] [6] [4]
- Press to calculate — [=]

And we have the return of $\log_4 64$ =3. Now if we observe carefully, we will find that

$\log_{10} 100 = \log_{10}(10^2)$ is equivalent to 2 while,

$\log_4 64 = \log_4(4^3)$ is equivalent to 3.

So what is $\log_5 25$ or $\log_{16} 4$? Check them out at in the calculator and see for yourself. In fact the calculator will return $\log_5 25$ =2 and $\log_{16} 4 = \frac{1}{2}$. This is the idea of the power rule. In general

♦ *For A > 0 and B any real number,* $\log_a A^B = B \cdot \log_a A$

### (ii) Special Properties Generated

What we have discussed before this actually can be re-expressed to generate more useful properties for problem solving.

Suppose we have $a^x = b^y$. From the definition of logarithm, we can express it into its logarithm equivalent, i.e. $\log_b(a^x) = y$. If we let *b* as *a* and *y* as *x* while satisfying the conditions of logarithms, we then have the following.

♦ $\log_a a^x = x$.

However if we substitute *x* in the property above as 1, then we have the following.

♦ $\log_a a^1 = 1$.

**<u>Example 4</u>**

Solve the equation $\log_{36}(6^x) = \frac{2}{3}$ for *x*.

 

**Solutions**

**[Operations]**

- Go to Computation mode — [MODE] [1]
- Enter $\log_{36}(6^x) = \frac{2}{3}$ — [log■□] [3] [6] [▶] [6] [x■] [ALPHA] [)]
- Continue entering — [▶] [▶] [ALPHA] [CALC] [■/□] [2] [▼] [3]
- Press to solve — [SHIFT] [CALC] (SOLVE) [1] [=]

And so we have the solution to $\log_{36}(6^x) = \frac{2}{3}$ as 1.33333. The exact solution is in fact $\frac{4}{3}$ and can be shown with the properties discussed.

$$\log_{36}(6^x) = \frac{2}{3} \Rightarrow x \cdot \log_{36} 6 = \frac{2}{3} \Rightarrow x \cdot \log_{36} 36^{\frac{1}{2}} = \frac{2}{3}$$

$$\Rightarrow \tfrac{1}{2} x \cdot \log_{36} 36 = \frac{2}{3} \Rightarrow \tfrac{1}{2} x = \frac{2}{3}$$

Therefore we have the solution of $x = \frac{4}{3}$.

One useful property we are yet to discuss is the changing of base. It essentially says that we can express the logarithm expression $\log_a A$ in base $b$ as $\frac{\log_b A}{\log_b a}$, provided that all unknowns mentioned observe the logarithm conditions. For example we can express $\log_9 17$ is indeed equal to $\frac{\log_{10} 17}{\log_{10} 9}$, and this is easily verified in the calculator.

**[Operations]**

- Go to Computation mode — [MODE] [1]
- Enter $\frac{\log_{10} 17}{\log_{10} 9} - \log_9 17$ — [■/□] [log] [1] [7] [)] [▼] [log] [9] [)]
- Continue entering — [▶] [−] [log■□] [9] [▶] [1] [7]
- Press to solve — [=]

The output of 0 indicates that $\frac{\log_{10} 17}{\log_{10} 9} - \log_9 17 = 0$ or $\frac{\log_{10} 17}{\log_{10} 9} = \log_9 17$.

More formerly, the base-change formula is as follow.

♦ If $a$, $b$ and $A > 0$ and $a, b \neq 1$ then $\log_a A = \frac{\log_b A}{\log_b a}$.

 

## EXERCISES

*Exercise 1*
Solve the following equations for *x*.

(i) $\log_3(7x+1) = 3$ (ii) $2\log_4 x^3 = 9$

(iii) $\log_{10}(100x^2) = \log_2 64$ (iv) $5^{2x+1} = 11^x$

*Exercise 2*
Solve $\log_2 x + \log_2(x+2) = \log_2(6x+1)$

## SOLUTIONS to Exercises

*Exercise 1*

(i) $x$ = 3.714285 or $\frac{26}{7}$ (ii) $x$ = 8

(iii) $x$ = 100, -100 (iv) $x$ = -1.960385

*Exercise 2*

$2+\sqrt{5}$

 