Modelling Exponential Decay

Barry Kissane
Murdoch University, WA
Australia

LEVEL
High school after students have learned about exponential functions.

OBJECTIVES
(i) To find an exponential decay model that fits a set of data
(ii) To use the model to make predictions

OVERVIEW
Exponential functions often provide good descriptions of natural growth and decay phenomena, so that exponential models are important tools for mathematical modelling. In the first activity, an exponential model is generated to describe the declining value of an asset. The second activity shows how the model developed can be used to make predictions. The exercises provide practice at using exponential functions to make predictions, as well as familiarising student with various aspects of calculator use, including Table mode and formula calculation.

EXPLORATORY ACTIVITIES
[Note]
(a) We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu, using
SHIFT MODE 1 (MthIO)

Exponential functions have the form $f(x) = AB^x$. When the function is increasing, exponential growth is involved. In this activity, we consider the case of a decreasing function, involving exponential decay. In this case, the value of B is less than 1, and it is more common to represent the function in the form $f(x) = AB^{-x}$, so that the value of B is still greater than 1.

Activity 1: Examining a depreciating asset
Some assets, such as motor vehicles, are valued at a smaller amount each year, when they are regularly used and also become less fashionable. Sally bought a new car in 2000, and every year checked the apparent value of her car by studying prices in used car yards. Of course, the prices vary a little, depending on the details of each car, so she estimated

an average amount each year that seemed to reflect the value of her car. Here are the results:

Year	2000	2001	2002	2003	2004	2005
Value ($)	12500	10400	8900	7800	6500	5500

To mathematically describe the declining value of her car, Sally noticed that the value each year seemed to be about the same proportion of the value in the previous year. For example, in the first year, she calculated as follows:

[Operations]

- New value — 1 0 4 0 0
- Divide by old value — ÷ 1 2 5 0 0
- Evaluate as decimal — SHIFT =

Sally interpreted the result of 0.832 to suggest that the value in 2001 was 83.2% of the value in 2000.

To compute the ratio for the second year, she used the percentage function directly, as follows:

[Operations]

- New value — 8 9 0 0
- Divide by old value — ÷ 1 0 4 0 0
- Find as percentage — SHIFT (
- Evaluate as decimal — SHIFT =

The car value in 2002 was 85.6% of the car value in 2001, a little different from the first depreciation, although about the same size.

Use the calculator to find the other three annual depreciations for yourself. Write down all the results and compare them with each other. You should find that each of them is approximately 85%.

So a suitable model for the value of Sally's car might be to assume that each year, the value is 85% of what it was the previous year, starting with a value of $12500. An exponential model to represent this is

$f(x) = 12500 \times 0.85^x$

where x refers to the number of years after 2000.

Another way to represent this model is to firstly represent 0.85 as 1.176^{-1} by finding the reciprocal of 0.85 on the calculator, as shown below.

[Operations]

- Find reciprocal — [0] [•] [8] [5] [x^{-1}]
- Evaluate as decimal — [SHIFT] [=]

So, 0.85 = 1.176^{-1}, approximately, and the exponential model can then be written in the standard form:

$f(x) = 12500 \text{x } 1.176^{-x}$

Although both models are the same (within rounding errors), the second version makes it clear that the value is decaying, and shows more clearly that *exponential decay* is involved. Models like this are sometimes also described as being *negative exponential* in form, because of the negative growth rate.

Activity 2: Studying and using the exponential model

To see how well, Sally's exponential model describe her data, use the Table mode of the calculator and evaluate the function for x = 0, 1, 2, … , 5.

[Operations]

- Select mode 7:Table — [MODE] [7]
- Enter function — [1] [2] [5] [0] [0]
- Enter function — [×] [1] [•] [1] [7] [6] [$x^{■}$] [(−)] [ALPHA] [)] [=]
- Start at 0 — [0] [=]
- End at 5 — [5] [=]
- Step is 1 — [1] [=]
- Explore table — [▶] [▼] [▼] [▼] [▼] [▼] [▼]
- When finished — [AC]

You should observe that the model fits the data quite well, as most of the model values are close to the actual data Sally collected.

To use the model to predict, press = to generate a fresh table, but use a higher end value (such as 10 or 20). Check for yourself that the model predicts that Sally's car will be worth only $2470 in 2010 and $488 in 2020. Such predictions are of course based on the assumption that the model continues to apply over that period, well past the data collected. In fact, in this case, it is very likely that Sally's car will no longer exist after 20 years and, if it does, may be worth even less than predicted.

Another way to make predictions with this exponential model allows you to merely enter a value for x and then generate the result, without having to generate an entire table.

The steps below show how to set up the calculator to do this. (Be aware that, since you must leave Table mode, the function is deleted from Table mode now.)

[Operations]

- Select mode 1:Comp [MODE] [1]
- Enter function [1] [2] [5] [0] [0]
- Enter function [×] [1] [•] [1] [7] [6] [$x^{\blacksquare}$] [(−)] [ALPHA] [)]

Then, to evaluate the function for any value of *x*, use the CALC command, enter the values of *x* and press the equals key. There are two examples shown below.

[Operations]

- Calculate [CALC]
- Enter 10 (for *x* = 10) [1] [0] [=]
- Calculate [CALC]
- Enter 20 (for *x* = 20) [2] [0] [=]

The results are the same as those in the table generated earlier. Try some more of these calculations for yourself with this exponential function.

Note that the function entered for calculation will be deleted when you change to a different mode, perform a different calculation or turn the calculator off.

Note also that there is another ways of studying exponential growth and decay on the calculator, using the statistical capabilities of the fx-911ES. These are explored in a different activity, called *Modelling with Exponential Functions*. You may wish to return to this activity related to Sally's car after you have completed the other activity, to compare the two procedures.

EXERCISES

The purpose of the exercises is to give you an opportunity to do some independent work, to develop fluency with calculator use and to learn more about exponential functions.

Exercise 1.
Use the percentage key on the calculator to express 4600 as a percentage of 5300.

Exercise 2
(a) Express the function $f(x) = 13(0.48)^x$ in the form of a negative exponential model, i.e., $f(x) = AB^{-x}$, with $B > 1$.
(b) Express the function $f(x) = 27(1.42)^{-x}$ in the form of an exponential model, i.e., $f(x) = AB^x$.

Exercise 3
(a) Enter the function $f(x) = 850(3.96)^{-x}$ into the calculator and then find the values of

$f(1)$, $f(4)$, $f(8)$and $f(10)$.
(b) Is this function increasing or is it decreasing? Explain your answer.

Exercise 4
Yuki purchased a new fishing boat in 1997, and every year obtained the value of the boat from the Insurance Company. Here are the results for the first few years:

Year	1997	1998	1999	2000	2001	2002
Value ($)	8000	6700	5050	4100	3300	2700

Construct an exponential model to describe the declining value of the boat. Use your model to estimate the value of the boat in 2005.

Exercise 5
Attempt this exercise only after completing the companion activity, *Modelling with Exponential Functions*, which uses the statistical capabilities of the calculator to build and use exponential models for data. Use an exponential regression model to model the value of Sally's car, and compare the results that you obtain with those in this activity.

SOLUTIONS

Exercise 1.
Use 4600/5300%= to get 86.8%

Exercise 2
(a) $0.48^{-1} = 2.083$, so $f(x) = 13(2.083)^{-x}$
(b) $1.42^{-1} = 0.704$, so $f(x) = 27(0.704)^{x}$

Exercise 3
$f(1) = 214.65$, $f(4) = 3.46$, $f(8) = 0.014$ and $f(10) = 0.0009$
(b) The function is decreasing, as larger x values lead to smaller values for $f(x)$.

Exercise 4
The value each year is about 82% of the value in the preceding year. So a suitable model is $f(x) = 8000 \times 0.82^{x} = 8000 \times 1.22^{-x}$, where x refers to the number of years after 1997. In 2005, $x = 8$, so the model predicts $8000 \times 1.22^{-8} = 1630$.

Exercise 5
The results are similar to those in Activity 1: the model is $f(x) = 12396(1.18)^{-x}$ and the predicted value for 2005 is $f(5) = 5532$.

