Solving Higher Order Equations

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LEVEL
High school after students have learned to solve cubic equations on the calculator

OBJECTIVES
(i) To understand the use of tabulation to solve equations
(ii) To understand the use of a solve command to solve higher order equations

OVERVIEW
Higher order equations occur frequently and students need to understand various solution methods. In this activity, we study in detail an example of a quartic equation. In the first activity, the solve command is used to look for solutions, but it is not clear how to find all solutions, or to decide how many solutions are present. In the second activity, the tabulation facility of the calculator is used, to search for solutions. In the third activity, the solve command is used again to refine these solutions. The exercises encourage students to understand the contributions of these two methods to solve higher order equations and also to develop expertise with calculator use, including use of the solve command and Table mode.

EXPLORATORY ACTIVITIES
[Note]
(a) We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu, using
[SHIFT] [MODE] [1] (MthIO)

Consider the following equation:

$8x^4 - 34x^3 + 31x^2 + 31x = 42$

This is a *quartic* equation as it is a polynomial for which the highest coefficient is 4. To solve an equation like this involves finding which values, if any, for *x* make the equation true.

You may recall that there is a standard formula for solving any quadratic equation and that the fx-991ES calculator has a procedure in EQN mode for solving any cubic equation. Furthermore, if a graphics calculator were available, we could get information about possible solutions to this quartic equation from a graph.

However, procedures of these three kinds are not available for quartic equations with this calculator, so that it is necessary to rely only on numerical methods. We will use the calculator to explore this equation in two related ways.

Activity 1: Using the solve command
A natural approach to solving an equation is to use the solve command, as was illustrated in the cubic equations activity you completed earlier. Indeed, this often provides a quick path to solutions, but it does not provide much information about the equation itself. In this case, the equation can be entered directly into the calculator, as shown below:

[Operations]

• Enter Computation mode	[MODE] [1]
• Enter the equation, $8x^4$	[8] [ALPHA] [)] [$x^■$] [4] [▶]
• Enter $-34x^3$	[−] [3] [4] [ALPHA] [)] [SHIFT] [x^2]
• Enter $+ 31x^2 + 31x$	[+] [3] [1] [ALPHA] [)] [x^2] [+] [3] [1] [ALPHA] [)]
• Enter the equals sign	[ALPHA] [CALC] (=)
• Enter 42	[4] [2]

Once the equation is entered, activate the solve command and choose a suitable initial guess. Since we don't really have any idea about possible solutions (except that there may be as many as four of them), a good way to start is to use a large negative value or a large positive value as an initial guess. The following steps show how to do this, choosing $x = -20$ (a fairly large negative number) as an initial guess.

[Operations]

• Enter Solver	[SHIFT] [CALC]
• Enter initial guess	[(−)] [2] [0]
• Solve equation	[=]
• Return to equation	[◀]

The calculator displays a solution at $x = -1$.

Check for yourself with similar steps to those above that using the solver with an initial guess of $x = 20$ (a fairly large positive number) also gives a solution, at $x = 2$.

There may be two more solutions, but it is difficult to tell in a systematic way; there may also be only one more solution or it is even possible that there are no more solutions.

One approach is to try another initial guess, between x = -1 and x = 2. A good choice would seem to be x = 1. Use this choice yourself, to see that another solution is found, with x = 1.5.

To see if there is a fourth solution, between x = -1 and x = 1, try the solve command with a starting value of x = 0. You will find that, once again, the solution x = 1.5 is found. Similarly, if an initial guess of x = -0.5 is chosen, the calculator returns the original solution of x = -1 again.

Perhaps an initial guess closer to x = 2 is needed? Try x = 1.9 for yourself, to see that the solve command merely reproduces the earlier solution of x = 2.

It is sometimes quite difficult to see how to proceed in situations like this, and it is not clear whether or not it is worth proceeding, as there may be only three solutions to the equation after all, and so no point in continuing the search.

Let us instead see how a solution process might proceed, using a table of values, as was illustrated in the cubic equation activity. At first, we will assume that the solve command has not yet been used.

Activity 2: Using a table
As for cubic equations, a solution method is to make and study a table for a suitable function. For example, the function $f(x) = 8x^4 - 34x^3 + 31x^2 + 31x$ could be tabulated and values of x which give the function a value near 42 can be found.

A better alternative is to use the function $f(x) = 8x^4 - 34x^3 + 31x^2 + 31x - 42$ and determine which values of x have $f(x)$ near zero. Any values of x for which the function has a zero will be solutions of the equation. The reason that this choice of function is easier than the first suggestion is that it is easier to find values in the table near zero, as they usually involve a change of sign.

The operations below show how to enter the function for tabulation.

[Operations]

• Enter Table mode	[MODE] [7]
• Enter the function, $8x^4$	[8] [ALPHA] [)] [$x^\blacksquare$] [4] [▶]
• Enter $-34x^3$	[−] [3] [4] [ALPHA] [)] [SHIFT] [x^2]
• Enter $+ 31x^2$	[+] [3] [1] [ALPHA] [)] [x^2]
• Enter $+ 31x - 42$	[+] [3] [1] [ALPHA] [)] [−] [4] [2]
• Press = to finish	[=]

The next step is to decide which values to tabulate. The fx-991ES allows for a maximum of 30 values to be tabulated, and you need to choose a start value for x, an end value for x and the increment (step) between values. At first, it is a good idea to choose a wide range, to see possible solutions (i.e. roots of the function, where $f(x)$ is zero), so we will use the interval from -10 to 10 in steps of 1. This interval is chosen partly because it includes both positive and negative values. It will also allow us to see the function values for larger positive and negative values of x. The calculator operations are shown below.

[Operations]

• Enter Start value	[(−)] [1] [0] [=]
• Enter End value	[1] [0] [=]
• Enter Step value	[1] [=]
• Explore the table of values	[▶] [▲] [▼] [▼] [▼] [▼] etc

It is important to study the table carefully to decide what to do next. The following observations are noteworthy in this case:

- There are two roots shown: x = -1 and x = 2
- For large negative x, values are very large. (eg $f(-10)$ = 116 748)
- For large positive x, values are also very large. (eg $f(10)$ = 49 368)
- For $-1 < x < 2$, values of $f(x)$ seem to be negative

The quartic function is dominated by the $8x^4$ term. As values of x get larger in either direction, it seems that this term will lead to very large positive terms and it is unlikely that solutions will be found that are greater than x = 10 or less than x = -10, as suggested by the table of values.

For a quartic equation, we expect four solutions, although they may not all be real (because some could be complex) and some may be repeated. So far, two solutions have been found. It seems that other solutions, if they exist, are likely to be in the interval (-1,2). So these would be good values to choose for our next interval.

A good choice of step would be 0.1, but this will generate a table with more than 30 values. Instead, we will choose the slightly smaller interval (-0.9,1.9), as shown below:

[Operations]

• Return to the function	[AC] [=]
• Enter Start value	[(−)] [0] [•] [9] [=]
• Enter End value	[1] [•] [9] [=]
• Enter Step value	[0] [•] [1] [=]
• Explore table of values	[▶] [▼] [▼] [▼] [▼] etc

Once again, it is important to study the table carefully, to decide what to do next. The following observations illustrate what kinds of study are necessary here.

- There is a root at x = 1.5
- Most function values are negative (but not all: $f(x) > 0$ for x = 1.6 and x = 1.7)

- The function changes sign from $x = 1.7$ to $x = 1.8$

These observations suggest that there is a fourth root between 1.7 and 1.8. The following steps show a suitable table generation process, with a step of 0.01 this time.

[Operations]

- Return to the function — [AC] [=]
- Enter Start value — [1] [•] [7] [=]
- Enter End value — [1] [•] [8] [=]
- Enter Step value — [0] [•] [0] [1] [=]
- Explore table of values — [▶] [▼] [▼] [▼] [▼] etc

The new table shows a root at $x = 1.75$, and thus another solution to the equation at this value.

So the table in this case has allowed for a more thorough understanding of this equation and an easier way to see that there are four solutions: $x = -1, 1.5, 1.75, 2$.

Activity 3: Using the solve command again
It is now clearer why the solve command was originally not successful. Some initial guesses between $x = -1$ and $x = 2$ produced the solution of $x = 1.5$, while others reproduced the solution of $x = -1$. Only a small range of starting values produce the solution at $x = 1.75$, and the earlier guesses were unfortunately not amongst them.

Enter the equation again and try the solve command again, this time with an initial guess of x = 1.8. The necessary operations are shown below:

[Operations]

- Enter Computation mode — [MODE] [1]
- Enter the equation, $8x^4$ — [8] [ALPHA] [)] [$x^{■}$] [4] [▶]
- Enter $-34x^3$ — [−] [3] [4] [ALPHA] [)] [SHIFT] [x^2]
- Enter $+ 31x^2 + 31x$ — [+] [3] [1] [ALPHA] [)] [x^2] [+] [3] [1] [ALPHA] [)]
- Enter the equals sign — [ALPHA] [=]
- Enter 42 — [4] [2]
- Enter Solver — [ALPHA] [CALC] (=)
- Enter initial guess — [1] [•] [8] [=]
- Solve equation — [=]

This time, the solve command generates the fourth (and final) solution of x = 1.75.

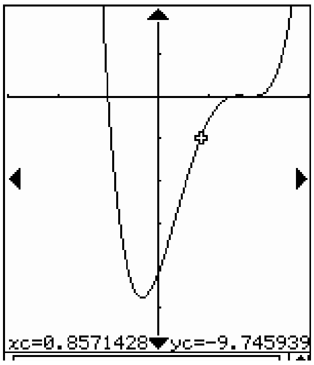
This example illustrates the value of developing expertise with both the solver and with the tabulation of functions to solve higher order equations. Although the calculator performs many computations, it is still necessary for the person using it to think carefully about the problem being addressed.

In practice, careful use of both methods together is a powerful means of solving equations, rather than relying on only one of the methods.

EXERCISES

The purpose of the exercises is to give you an opportunity to do some independent work, to develop fluency with calculator use and to learn more about solving cubic equations.

Exercise 1
Here is a graph of the function $f(x) = 8x^4 - 34x^3 + 31x^2 + 31x - 42$, which was used in the activities. (The graph was drawn on the Casio *ClassPad 300.*) A graph was not available when considering the equation; if it had been available, what information could you have obtained from it to help the solution process?



Exercise 2
Solve the quartic equation $2x^4 + 7x^3 - 48x^2 + 17x + 70 = 0$.

Exercise 3
Solve the quartic equation $2x^4 + 15x^3 + 18x^2 = 64x + 96$.

Exercise 4
Solve the quartic equation $z^4 = 7z^2 - 4z + 5$, giving each solution correct to two decimal places.

Exercise 5
Write down all the solutions to the quartic equation:

$(t + 1)(t - 2)(t + 7)(2t - 5) = 0$

Check your solutions using the solve command.

Exercise 6
Write down an equation with solutions x = 1, 5, -1 and 3. Then use the solve command to verify that the equation does in fact have these solutions.

Exercise7
A polynomial equation with highest term a fifth power is called a *quintic* equation. In the same way that quartic equations have a maximum of five real solutions, quintic equations have a maximum of five real solutions. Consider the following quintic equation:

$x^5 - 5x^4 + 5x^3 - 25x^2 + 4x = 20$

(a) Find all real solutions to the equation
(b) Check by substitution that $x = i$ is a complex solution to the equation.

SOLUTIONS

Exercise 1
A graph would have helped with a choice of starting values for the solve command and would also have made clear how many solutions there are to the equation.

Exercise 2
Solutions are x = -7, -1, 2, 2.5

Exercise 3
Solutions are x = -4,-1.5, 2

Exercise 4
Solutions are z = -2.79, -0.62, 1.62, 1.79 (It is necessary to use x as the variable in the calculator solution.)

Exercise 5
Solutions are t = -1, 2, -7, 5/2.

Exercise 6
One possibility is $(x - 1)(x - 5)(x + 1)(x - 3) = 0$.

Exercise7
The only real solution is $x = 5$.

